Lecture 5 18-09-17

**Def:** $2^{nd}$ order Linear ODE:

$$\boxed{y'' + p(t)y' + q(t)y = r(t)}$$

for $t \in I = (a,b)$

- It is called homogeneous if $r(t) \equiv 0$

**Def:** (IVP for $2^{nd}$ order)

$$(*) \text{---} \begin{cases} y'' + p(t)y' + q(t)y = r(t) \\ y(t_0) = x_0, \quad y'(t_1) = x_1 \end{cases}$$

for $t_0, t_1 \in I$.

(we will take $t_0 = t_1$ for most of our discussion)

**Thm:** (Existence and uniqueness)

For the above IVP $(*)$, $\exists!$ solution $y(t)$ defined on the whole $I$

Pf: (Omitted, see the Reference book)

**Example:** If take $x_0 = x_1 = 0$, $\Rightarrow \exists!$ sol. $y(t) \equiv 0$

## § Superposition principle:

**Thm:** • If $y_1(t)$, $y_2(t)$ two solutions to

$$y'' + p(t)y' + q(t) = 0$$

then $c_1 y_1(t) + c_2 y_2(t)$ is also a solution

**Pf:** Write

$$
\begin{aligned}
q(t)\,y(t) &= q(t)\left(c_1 y_1(t) + c_2 y_2(t)\right)\\
+\; p(t)\,y'(t) &= p(t)\left(c_1 y_1'(t) + c_2 y_2'(t)\right)\\
+\; y''(t) &= c_1 y_1''(t) + c_2 y_2''(t)
\end{aligned}
$$

$$
\begin{aligned}
y'' + p(t)y' + q(t)y &= c_1\left(y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\right)\\
&\quad + c_2\left(y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\right)\\
&= 0
\end{aligned}
$$

**Moral:** • Let $y_1, y_2$ be sol to the homogeneous eq

$$\mathcal{S} := \{y = c_1 y_1 + c_2 y_2 \;,\; c_1, c_2 \in \mathbb{R}\}$$

• Consider the IVP (*) in the previous page

want to look for $y \in \mathcal{S}$ s.t.

$$y(t_0) = x_0, \quad y'(t_0) = x_1$$

i.e. to find $c_1, c_2$ s.t

$$c_1 y_1(t_0) + c_2 y_2(t_0) = x_0$$

$$c_2 y_1'(t_0) + c_2 y_2'(t_0) = x_1$$

written as matrix:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

idea: If we can invert the matrix $\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$, then we can solve the IVP.

Def: We let the Wronskian

$$W(y_1, y_2)(t) := \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

$$= y_1(t)\, y_2'(t) - y_1'(t)\, y_2(t)$$

associated to two solutions $y_1(t)$, $y_2(t)$.

Thm: • If $W(y_1, y_2)(t_0) \neq 0$ at a point $t_0 \in I$

i.e. the solution $y(t)$ to any IVP is in $\mathcal{S}$

(exist and unique by the Existence & Uniqueness thm)

• The space of general solution

$$\{ y \mid y'' + p(t)y' + q(t)y = 0 \} = \mathcal{S}$$

is a 2-dimensional vector space with basis $y_1, y_2$.

Pf: • Let $\tilde{\mathcal{S}} := \{ y \mid y'' + p(t)y' + q(t)y = 0 \}$

from the superposition principle, we have

$\tilde{\mathcal{S}}$ is a vector space.

• We have $\mathcal{S} \subseteq \tilde{\mathcal{S}}$, and if we take any $y \in \tilde{\mathcal{S}}$, we let $y(t_0) = x_0$, $y'(t_0) = x_1$

$\Rightarrow$ $y$ is a sol to the above IVP

$\Rightarrow$ $y \in \mathcal{S}$ and hence $\mathcal{S} = \tilde{\mathcal{S}}$.

- We have to show $y_1, y_2$ linear independent:

if $c_1 y_1(t) + c_2 y_2(t) = 0$

$\Rightarrow c_1 y_1'(t) + c_2 y_2'(t) = 0$

at $t = t_0$: $\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$

since $W(y_1, y_2)(t_0) \neq 0 \Rightarrow$ the above matrix in the L.H.S. is invertible.

$\Rightarrow c_1 = 0, \quad c_2 = 0$

Prop: On an interval $I = (a,b)$, if $y_1, y_2$ be two solutions to $y'' + p(t)y' + q(t) = 0$,

if the Wronskian $W(y_1, y_2)(t_0) \neq 0$, then

$W(y_1, y_2)(t) \neq 0$ for any $t \in (a,b)$.

Pf: If we have $\hat{t}$ s.t. $W(y_1, y_2)(\hat{t}) = 0$

then we consider the IVP at $\hat{t}$:

$y(\hat{t}) = \hat{x}_0, \quad y'(\hat{t}) = \hat{x}_1$

$W(y_1, y_2)(\hat{t}) = 0 \Rightarrow \begin{pmatrix} y_1(\hat{t}) \,, & y_2(\hat{t}) \\ y_1'(\hat{t}) \,, & y_2'(\hat{t}) \end{pmatrix}$

is <u>NOT</u> invertible and hence has rank $\leq 1$

$\Rightarrow \exists \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$ s.t.

$$\begin{pmatrix} y_1(\hat{t}) & y_2(\hat{t}) \\ y_1'(\hat{t}) & y_2'(\hat{t}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \end{pmatrix} \text{ has}$$

<u>No</u> solution.

However, if we take a sol $\tilde{y}$ to the IVP at $\hat{t}$ (such a solution exist by the existence and uniqueness theorem)

consider $\tilde{y}(t_0) = x_0$, $\tilde{y}'(t_0) = x_1$.

by $W(y_1, y_2)(t_0) \neq 0$

$\Rightarrow \exists\, c_1, c_2$ s.t. $\tilde{y} = c_1 y_1 + c_2 y_2$

$$\Rightarrow \begin{pmatrix} y_1(\hat{t}) \,, & y_2(\hat{t}) \\ y_1'(\hat{t}) \,, & y_2'(\hat{t}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \end{pmatrix}$$

$\longrightarrow \longleftarrow$

**Def:** $y_1, y_2$ is called a <u>fundamental set</u> of solution if they satisfy $y'' + p(t)y' + q(t)y = 0$ <u>and</u> $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$.

**Example:** Consider

$$y'' + \underbrace{\frac{3}{2t}}_{=p(t)} y' \overbrace{- \frac{1}{2t^2}}^{=q(t).} y = 0, \quad \text{on } I = (0, +\infty)$$

Let $y_1 = t^{1/2}$, $y_2 = t^{-1}$ both of them are solution to the above equation, we compute

$$y_1' = \frac{1}{2} t^{-\frac{1}{2}}, \quad y_2' = -t^{-2}$$

$$W(y_1, y_2)(t) = \det \begin{pmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{pmatrix} = -t^{-3/2} - \frac{1}{2} t^{-3/2}$$

$$= -\frac{3}{2} t^{-3/2}$$

We have $W(y_1, y_2)(t) \neq 0$ for $t \in (0, +\infty)$

$\Rightarrow$ General solution:

$$y(t) = c_1 t^{-1/2} + c_2 t^{-1}.$$

**Thm** (Abel's thm)

If $y_1, y_2$ solution to the equation

$$y'' + p(t) y' + q(t) y = 0,$$

then the Wronskian $W(t) = W(y_1, y_2)(t)$

$$\boxed{W(t)' + p(t) W(t) = 0}$$

$$\Rightarrow \quad W(t) = C \exp\left(-\int p(t)\,dt\right).$$

where $C$ depends on $y_1, y_2$ but NOT $t$.

Pf: $W(t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$

$$\Rightarrow W'(t) = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$$

$$= y_1 y_2'' - y_1'' y_2$$

$$= y_1(-p(t) y_2' - q(t) y_2) - y_2(-p(t) y_1' - q(t) y_1)$$

$$= -p(t)(y_1 y_2' - y_2 y_1') = -p(t) W(t)$$

$\Rightarrow W(t) = C \exp\left(-\int_{t_0}^{t} p(s)\,ds\right)$, with

$C = W(t_0)$

$\square$

**Rk:** We have shown earlier that either $W(y_1,y_2)(t) \equiv 0$ or $W(y_1,y_2)(t) \neq 0 \ \forall t$

This can be seen by Abel's thm:

$\because \ W(y_1,y_2)(\hat{t}) = 0 \Rightarrow c = 0$

$\Rightarrow W(y_1,y_2)(t) \equiv 0.$

**Thm:** To summarize, TFAE:

1. $y_1, y_2$ linearly independent.
2. $W(y_1,y_2)(t_0) \neq 0$ for some point $t_0 \in I$
3. $W(y_1,y_2)(t) \neq 0 \quad \forall t \in I.$

**Pf:**

- We have shown $(2) \Longleftrightarrow (3)$ and $(2) \Rightarrow (1)$.
- It remains to show $(1) \Rightarrow (2)$

suppose we have

$$W(y_1,y_2)(t_0) = 0$$

$$\Rightarrow \det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = 0$$

$\rightarrow$ $\begin{pmatrix} y_1(t_0) \\ y_1'(t_0) \end{pmatrix}$ and $\begin{pmatrix} y_2(t_0) \\ y_2'(t_0) \end{pmatrix}$ are linearly dependent as vectors.

By linear algebra:

$\exists$ $c_1, c_2$ NOT both zero s.t.

$$c_1 \begin{pmatrix} y_1(t_0) \\ y_1'(t_0) \end{pmatrix} + c_2 \begin{pmatrix} y_2(t_0) \\ y_2'(t_0) \end{pmatrix} = 0.$$

Suppose we let

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$\Rightarrow \quad y(t_0) = 0, \quad y'(t_0) = 0$$

By uniqueness $\Rightarrow$ $y(t) \equiv 0$

$\Rightarrow$ $c_1 y_1(t) + c_2 y_2(t) \equiv 0$

$\Rightarrow$ $c_1 = 0, \ c_2 = 0$

$\longrightarrow \leftarrow$ $\boxtimes$

**Example:** Assume we have

$$y'' + p(t) y' + q(t) y = 0$$

and we are given a solution $y_1 \neq 0$.

How we look for another sol $y_2$ s.t. $y_1, y_2$ is a fundamental set?

Ans: We let $W(t) = \exp(-\int p(t)dt)$

- And we solve for $z(t)$ satisfying

$$y_1(t)z' - zy_1'(t) = W(t)$$

be a $1^{st}$-order ODE

- Suppose $y_1(t) \neq 0$ on $I$

$\Rightarrow$ $z(t)$ exist on $I$

then we have

$$y_1'z' + y_1z'' - z'y_1' - zy_1'' = W'$$

$$y_1z'' + z(p(t)y_1' + q(t)y_1) = -p(t)W$$

$$y_1z'' + p(t)(zy_1' + W) + zq(t)y_1 = 0$$

$$\Rightarrow y_1z'' + y_1p(t)z' + y_1q(t)z = 0.$$

$$\Rightarrow z'' + p(t)z' + q(t)z = 0.$$

- $W(y_1, z)(t) = \exp(-\int p(t)dt) \neq 0.$

Example: • Consider for $t>0$

$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$, suppose we know

$y_1(t) = t^{1/2}$ is a solution.

we choose the constant $c \neq 0$

• Let $W(t) = \exp\left(-\int \frac{3}{2t}\,dt\right) = t^{-3/2}$

We solve

$$t^{1/2} z' - \frac{1}{2} t^{-1/2} z = t^{-3/2}$$

$$\Rightarrow \quad z' - \frac{1}{2t} z = t^{-2}$$

Again, let $\mu = \exp\left(\int \frac{1}{2t}\,dt\right) = t^{1/2}$.

$$\Rightarrow \quad z = t^{-1/2}\left[\int t^{-3/2}\,dt + \cancel{c}\right]$$

we just need a solution

$$= -2t^{-1}$$

There for $y_1(t) = t^{1/2}$, $y_2(t) = -2t^{-1}$ is a fundamental set of solution.

## To solve an IVP:

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = x_0, \quad y'(t_0) = x_1 \end{cases}$$

**Step 1:** Find a fundamental set of solution $y_1, y_2$.

**Step 2:** Solve the matrix equation

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

**Step 3:** Let $y = c_1 y_1 + c_2 y_2$.

## Main question:

How to find $y_1, y_2$? (or at least $y_1$?)

To do that in general: in next lecture we consider

**Def:** (homogeneous eqt with constant coefficient).

$$\boxed{y'' + by' + cy = 0}$$

constant